*Projective Differential Geometry of One-parameter Families of Space Curves, and Conjugate Nets on a Curved Surface.**

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Introduction.

In studying a geometric configuration, it is often convenient to consider the configuration as part of another, which is characterized by some peculiar geometric property. A one-parameter family of curves on a surface, for instance, is best studied—or so, at least, it seems to the writer—by considering it in connection with the family conjugate thereto. This is, in fact, the point of view of the present paper, in which a one-parameter family of curves is investigated as a component family of a conjugate net.

There seems, at first sight, to be a certain loss of generality in adopting this procedure, since the determination of the family of curves conjugate to a given family requires the integration of a partial differential equation of the first order, an integration which it is, in general, impossible to perform explicitly. It is shown, however, that the restriction is only an apparent one; the possibility of removing it results from the unique analytic apparatus employed. The one-parameter family of curves is defined by a fundamental system of solutions of a completely integrable system of partial differential equations. To Wilczynski is due the credit of recognizing the importance of completely integrable systems in projective differential geometry. By the use of such a system conjugate nets appear in a new light. Darboux's classic treatment depends upon a single partial differential equation; we have associated therewith a second equation, forming with the first a completely integrable system, and are thus enabled to found a purely projective theory of conjugate nets.

The present paper is little more than an introduction to the subject; we have reserved for another occasion the treatment of various interesting topics, in particular the study of certain congruences associated with the configuration.

§ 1. *The Problem, and the System of Differential Equations.*

Let the homogeneous coordinates $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$ of a point in space be given by the equations

* Read before the American Mathematical Society, October 25, 1913.

$$y^{(k)} = f^{(k)}(u, v), \qquad (k = 1, 2, 3, 4). \tag{1}$$

These equations define a surface, referred to the parametric curves $u =$ const., $v =$ const. It is our purpose to study the single family of curves $v =$ const., which we shall denote by C_u. The method employed will be that of Wilczynski, in which a projective theory consists in the discussion of the invariants and covariants, under suitable transformations, of a completely integrable system of differential equations.

The curves $u =$ const., or in our notation the curves C_v, may be asymptotic curves on the surface defined by equations (1). If such be the case, we may always make a proper transformation of the form

$$\bar{u} = U(u, v), \quad \bar{v} = V(v),$$

which alters the parametric curves C_v, but leaves unchanged the curves C_u; in this way we may always select for C_v a set of curves which are not asymptotic, except in the trivial case where all curves are asymptotic, *i. e.*, the surface is a plane.* We shall suppose, then, that the curves C_v are not asymptotic curves. It follows that the functions (1) can not be solutions of a differential equation of the form†

$$\alpha y_{vv} + \beta y_u + \gamma y_v + \delta y = 0. \tag{2}$$

In other words, the four functions (1) can not each satisfy identically the same relation of form (2); hence, *the determinant*

$$D = \begin{vmatrix} y_{vv}^{(1)} & y_u^{(1)} & y_v^{(1)} & y^{(1)} \\ y_{vv}^{(2)} & y_u^{(2)} & y_v^{(2)} & y^{(2)} \\ y_{vv}^{(3)} & y_u^{(3)} & y_v^{(3)} & y^{(3)} \\ y_{vv}^{(4)} & y_u^{(4)} & y_v^{(4)} & y^{(4)} \end{vmatrix} \tag{3}$$

is not identically zero.

Consider the two partial differential equations

$$\left.\begin{aligned} y_{uu} &= a\,y_{vv} + b\,y_u + c\,y_v + d\,y, \\ y_{uv} &= a'\,y_{vv} + b'\,y_u + c'\,y_v + d'\,y. \end{aligned}\right\} \tag{4}$$

The coefficients in these equations may be so determined that the functions (1) will be solutions of system (4). In fact, by substituting successively in the first of equations (4) the quantities $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$, we obtain four equations which, since the determinant D is not zero, may be solved for a, b, c, d. Similarly, we may calculate a', b', c', d'.

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* Systems of curves in the plane have already been studied from the standpoint of projective differential geometry by Wilczynski, "One-parameter Families and Nets of Plane Curves," *Transactions of the American Mathematical Society*, Vol. XII (1911); also by the author in a paper entitled "One-parameter Families of Curves in the Plane," *ibid.*, Vol. XV (1914).

† Darboux, "Théorie Générale des Surfaces," Vol. I, p. 144.

The system (4) thus set up will be completely integrable, unless, as we shall see, $a'^2 - a = 0$. To make the discussion perfectly general, let us suppose given any system of differential equations, of form (4), in which the coefficients are perfectly arbitrary. In order that the system be completely integrable, with just four fundamental solutions, it is necessary that all derivatives of y be expressible linearly, in a unique way, in terms of the same four quantities. The derivatives y_{uu}, y_{uv} are so expressed in terms of y_{vv}, y_u, y_v, y; let us see if it is the same with the higher derivatives of y. Differentiating each of equations (4) with respect to u and v, we obtain the four equations:

$$\left.\begin{aligned}
y_{uuu} - a\, y_{uvv} &= b\, y_{uu} + c\, y_{uv} + a_u y_{vv} + (b_u + d)\, y_u + c_u y_v + d_u y,\\
y_{uuv} - a\, y_{vvv} &= b\, y_{uv} + (c + a_v)\, y_{vv} + b_v y_u + (c_v + d)\, y_v + d_v y,\\
y_{uuv} - a'\, y_{uvv} &= b'\, y_{uu} + c'\, y_{uv} + a'_u y_{vv} + (b'_u + d')\, y_u + c'_u y_v + d'_u y,\\
y_{uvv} - a'\, y_{vvv} &= b'\, y_{uv} + (c' + a'_v)\, y_{vv} + b'_v y_u + (c'_v + d')\, y_v + d'_v y.
\end{aligned}\right\} \qquad (5)$$

We may replace y_{uu}, y_{uv} in these equations by their equivalents from (4); the system may then be solved for y_{uuu}, y_{uuv}, y_{uvv}, y_{vvv} linearly in terms of y_{vv}, y_u, y_v, y, provided the determinant of the coefficients on the left,

$$\begin{vmatrix} 1 & 0 & -a & 0 \\ 0 & 1 & 0 & -a \\ 0 & 1 & -a' & 0 \\ 0 & 0 & 1 & -a' \end{vmatrix} = a'^2 - a,$$

is not zero. We shall suppose that $a'^2 - a \neq 0$, leaving for a time the geometric interpretation of the excepted cases.

To find the fourth derivatives, we differentiate equations (5) with respect to u, and also with respect to v, thus obtaining eight equations. But differentiation of the first and third of (5) with respect to v gives the same equations as differentiation of the second and fourth with respect to u. It follows that there are just six equations to determine the five fourth derivatives of y. One of the fourth derivatives may therefore be calculated in two ways; but the two expressions must be identically equal, in order that the system (4) be completely integrable. Equating these two expressions, which we suppose expressed (linearly) in terms of y_{vv}, y_u, y_v, y, we obtain a relation of form (2), viz.,

$$\alpha\, y_{vv} + \beta\, y_u + \gamma\, y_v + \delta\, y,$$

which is to be satisfied identically. But we have assumed the curves C_v to be other than asymptotic, so we must have $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$. We thus obtain four equations connecting the coefficients of (4) and their first and second derivatives. If these *conditions of complete integrability* be satisfied, the fourth derivatives of y will be expressible uniquely in terms of y_{vv}, y_u, y_v, y. The higher derivatives will then also be uniquely expressible; in fact, further differentiation of (5) will always give just enough new equations to determine

the derivatives of next higher order. We shall not write out until later the integrability conditions $\alpha=\beta=\gamma=\delta=0$.

System (4) is therefore completely integrable, with four fundamental solutions, provided $a'^2-a\neq 0$ and the integrability conditions are satisfied. If equations (1), viz.,

$$y^{(k)}=f^{(k)}(u,v), \qquad (k=1,2,3,4),$$

be a fundamental system of solutions of (4), then any fundamental system of solutions will be given by

$$\bar{y}^{(k)}=c_{k1}\,y^{(1)}+c_{k2}\,y^{(2)}+c_{k3}\,y^{(3)}+c_{k4}\,y^{(4)}, \qquad (k=1,2,3,4),$$

where the c_{kl}'s are constants such that their determinant $|c_{kl}|\neq 0$. In geometric language, *the configuration represented by any fundamental system of solutions of* (4) *is a projective transformation of the configuration represented by any other fundamental system of solutions.*

In the present paper we shall deal exclusively with a completely integrable system of form (4), for which of course $a'^2-a\neq 0$. If, however, for a system of form (4), $a'^2-a=0$, the third derivatives can not be calculated uniquely from (5). Two cases may arise; either (a) the equations (5) are incompatible, in which case there are fewer than three fundamental solutions of system (4), or (b) one equation of system (5) is a consequence of the other three. In this case one of the third derivatives is perfectly arbitrary, so that there are an infinite number of fundamental solutions of (4). Evidently, neither of these cases can yield a projective theory in three-dimensional space, because for such a theory we need four fundamental solutions. For case (a), it may be shown that a system of four solutions represents geometrically a plane or line.* The curves C_u, which we wish to study, can not lie on a line, but they may lie in a plane. If, now, we associate with system (4) a third differential equation, of form (2), the resulting system will be completely integrable if certain new integrability conditions be satisfied. The completely integrable system will have, not four, but three fundamental solutions. Such a system has indeed been studied by Wilczynski in his memoir, "One-parameter Families and Nets of Plane Curves."† There is no necessity therefore for considering the problem here.

In case (b), one of the third derivatives is arbitrary. This is the *involutory* case, and a system of four solutions represents either a curve or a developable surface.‡ The first of these is of no interest to us; but the second is

* E. J. Wilczynski, first memoir on curved surfaces, *Transactions of the American Mathematical Society*, Vol. VIII (1907).

† *Transactions of the American Mathematical Society*, Vol. XII (1911).

‡ E. J. Wilczynski, first memoir on curved surfaces, *Transactions of the American Mathematical Society*, Vol. VIII (1907).

important. System (4) by itself is in this case not completely integrable; but by associating with it a differential equation of the third order, independent of equation (5), all the third-order derivatives are calculable in terms of y_{vv}, y_u, y_v, y. If, further, integrability conditions be satisfied for the new system of differential equations, that system will be completely integrable, with four fundamental solutions. Such a system would be suitable for studying systems of curves on a developable surface; in fact, a projective theory of developable surfaces has recently been constructed on the basis of such a system of differential equations.* *We shall suppose in this paper that the system of curves C_u does not lie on a developable surface.*

We shall make a further restriction on the curves C_u, the reason for which will become apparent in the sequel. We suppose that the curves C_u are not asymptotic curves on the surface (1), so that† $a \neq 0$. To study a system of asymptotic curves would require a procedure different from that which we propose to follow; Wilczynski has already studied surfaces referred to their asymptotic curves,‡ so that we may well leave aside such systems. *The systems of ∞^1 curves C_u which we do not study in the present paper are those which either lie on a developable (including the case of a plane) or are asymptotic curves on the surface which they determine. In the system of differential equations* (4) *we therefore suppose that*

$$a'^2 - a \neq 0, \tag{6}$$

$$a \neq 0. \tag{7}$$

§ 2. *Intermediate Form of the System of Differential Equations.*

The most general transformations of the variables which leave fixed the system of curves C_u are

$$y = \lambda \bar{y}, \tag{8}$$

where λ is any function of u and v, and

$$\bar{u} = U(u, v), \quad \bar{v} = V(v). \tag{9}$$

These transformations disturb any property not connected intrinsically with the system of curves C_u. The system of differential equations (4) has its general form unchanged when subjected to the transformations (8) and (9), but the coefficients of the system are altered. Since the system is characteristic, not only of the single geometric configuration represented by a funda-

* W. W. Denton, "Projective Differential Geometry of Developable Surfaces," *Transactions of the American Mathematical Society*, Vol. XIV (1913).

† Darboux, "Théorie Générale des Surfaces," t. I, p. 144.

‡ First three of the five memoirs on curved surfaces, *Transactions of the American Mathematical Society*, Vols. VIII–X (1907–09).

mental system of solutions, but also of all its projective transformations, it follows that any property, whose expression in terms of the differential equations remains unchanged under the transformations (8) and (9), is characteristic of the curves C_u and their projective transformations; *i. e.*, it is a projective property. Accordingly, *we call invariants those functions of the coefficients of system* (4) *and their derivatives which remain unchanged, except for a factor, when transformed by* (8) *and* (9); *if the variable y and its derivatives also appear explicitly in such an invariant function, we call the function a covariant.* If similar functions remain unchanged under (8), but not necessarily under (9), we shall call them *seminvariants* and *semi-covariants.*

It will simplify matters to introduce an intermediate form for the system of differential equations. To do this, we make the transformation

$$\bar{u} = U(u, v), \quad \bar{v} = v, \tag{10}$$

which is a subgroup of (9). Denoting $\partial y/\partial \bar{u}$, $\partial y/\partial \bar{v}$, etc., by $\bar{y}_u$, $\bar{y}_v$, etc., we find

$$\left.\begin{aligned} y_u &= \bar{y}_u U_u, \quad y_v = \bar{y}_u U_v + \bar{y}_v, \\ y_{uu} &= \bar{y}_{uu} U_u^2 + \bar{y}_u U_{uu}, \\ y_{uv} &= \bar{y}_{uu} U_u U_v + \bar{y}_{uv} U_u + \bar{y}_u U_{uv}, \\ y_{vv} &= \bar{y}_{uu} U_v^2 + 2\,\bar{y}_{uv} U_v + \bar{y}_{vv} + \bar{y}_u U_{vv}. \end{aligned}\right\} \tag{11}$$

Substitute these in (4), obtaining a system of two partial differential equations. These may be solved for $\bar{y}_{uu}$ and $\bar{y}_{uv}$, and the resulting system will be precisely of the form (4). Denoting its coefficients by $\bar{a}$, $\bar{b}$, ..., $\bar{a}'$, $\bar{b}'$, ..., we easily find for them the following expressions, in which we have written

$$\Delta = U_u (U_u^2 - 2\,a' U_u U_v + a\,U_v^2), \tag{12}$$

and performed no divisions:

$$\left.\begin{aligned} \Delta\,\bar{a} &= a\,U_u, \\ \Delta\,\bar{b} &= a\,U_u U_{vv} - 2\,a\,U_v U_{uv} - (U_u - 2\,a' U_v)\,U_{uu} \\ &\quad + [b\,(U_u - 2\,a' U_v) + 2\,a\,b' U_v]\,U_u \\ &\quad + [c\,(U_u - 2\,a' U_v) + 2\,a\,c' U_v]\,U_v, \\ \Delta\,\bar{c} &= c\,(U_u - 2\,a' U_v) + 2\,a\,c' U_v, \\ \Delta\,\bar{d} &= d\,(U_u - 2\,a' U_v) + 2\,a\,d' U_v, \\ \Delta\,\bar{a}' &= U_u\,(a' U_u - a\,U_v), \\ \Delta\,\bar{b}' &= (U_u U_v - a' U_v^2)\,U_{uu} - (U_u^2 - a\,U_v^2)\,U_{uv} + U_v\,(a' U_u - a\,U_v)\,U_{vv} \\ &\quad + [b'\,(U_u^2 - a\,U_v^2) - b\,(U_u U_v - a' U_v^2)]\,U_u \\ &\quad + [c'\,(U_u^2 - a\,U_v^2) - c\,(U_u U_v - a' U_v^2)]\,U_v, \\ \Delta\,\bar{c}' &= c'\,(U_u^2 - a\,U_v^2) - c\,(U_u U_v - a' U_v^2), \\ \Delta\,\bar{d}' &= d'\,(U_u^2 - a\,U_v^2) - d\,(U_u U_v - a' U_v^2). \end{aligned}\right\} \tag{13}$$

We may make $\bar{a}'$ vanish by choosing the function $U(u, v)$ as a solution of the differential equation

$$a' U_u - a U_v = 0. \tag{14}$$

It is easy to show that with this choice of U the expression Δ can not be zero. In fact, it takes the form, since $a \neq 0$,

$$\Delta = U_u^3 (a - a'^2)/a. \tag{15}$$

But U_u can not be zero for a proper transformation, and moreover $a - a'^2 \neq 0$, by (7). We may therefore reduce the system (4) to one for which $\bar{a}' = 0$, by solving equation (14).

The geometric interpretation of this transformation is very simple. If in the second of equations (4) we put $a' = 0$, we see from its form that the surface (1) will be referred to the family of curves C_u and to the family of curves conjugate to C_u.* We may therefore state the result:

By the integration of the partial differential equation

$$a' U_u - a U_v = 0, \tag{14}$$

we may always determine the system of curves conjugate to a system of curves C_u *which lie on a non-developable surface and are not asymptotic. The system of differential equations* (4) *will thereby be thrown into the form*

$$\left.\begin{aligned} y_{uu} &= a y_{vv} + b y_u + c y_v + d y, \\ y_{uv} &= b' y_u + c' y_v + d' y. \end{aligned}\right\} \tag{16}$$

We call this the *intermediate form* of the system of differential equations.

It is easily seen that *the most general transformation of the independent variables which leaves fixed the system of curves* C_u *and its conjugate system* C_v *is of the form*

$$\bar{u} = \phi(u), \quad \bar{v} = \psi(v), \tag{17}$$

where ϕ *and* ψ *are arbitrary.* This is of course the most general transformation of the independent variables which leaves the system (16) in the intermediate form (characterized by the condition $a' = 0$), without interchanging the parameters u and v.

We shall accordingly study the system (16) under the transformations (8) and (17). Our theory then becomes equivalent to that of a surface referred to a conjugate net as parameter curves. Yet we may legitimately call it the theory of the single system of curves C_u, because the conjugate system C_v is uniquely determined by the system C_u. Moreover, we shall find it possible to express the invariants and covariants of (16), under the transformations (8) and (17), in terms of the coefficients of the original system of differential equations (4), *without requiring the actual integration of the partial differential equation* (14).

* Darboux, "Théorie Générale des Surfaces," t. I, p. 122.

We may use equation (14) to express the coefficients of the intermediate form in terms of the original quantities a, b, c, etc. Differentiation of (14) with respect to u and with respect to v gives the two relations

$$\left.\begin{aligned} a' U_{uu} - a U_{uv} + a'_u U_u - a_u U_v = 0, \\ a' U_{uv} - a U_{vv} + a'_v U_u - a_v U_v = 0. \end{aligned}\right\} \qquad (18)$$

Using these and equation (14), we may remove from (13) all derivatives of U except U_u and U_{uu}. The result is:

$$\left.\begin{aligned} &\bar{a} = a^2 / U_u^2 (a - a'^2), \\ &\bar{b} = [a\,(a a'_v - a' a_v) - a'\,(a a'_u - a' a_u) + b\,(a^2 - 2 a a'^2) + 2 a^2 a' b' \\ &\qquad + c\,(a a' - 2 a'^3) + 2 a a'^2 c'] / U_u a\,(a - a'^2) - U_{uu} / U_u^2, \\ &\bar{c} = a\,[c\,(a - 2 a'^2) + 2 a a' c'] / U_u^2\,(a - a'^2), \\ &\bar{d} = a\,[d\,(a - 2 a'^2) + 2 a a' c'] / U_u^2\,(a - a'^2), \\ &\bar{b}' = [a\,(a b' - a' b) + a'\,(a c' - a' c) + a' a_u - a a'_u] / a^2, \\ &\bar{c}' = (a c' - a' c) / a U_u, \quad \bar{d}' = (a d' - a' d) / a U_u. \end{aligned}\right\} \qquad (19)$$

It will be noticed that $\bar{b}$ is the only coefficient in which U_{uu} occurs.

§ 3. *The Integrability Conditions for the System in its Intermediate Form.*

We proceed to calculate the integrability conditions for the system in its intermediate form. Specializing equations (5) by putting $a' = 0$, solving for y_{uuu}, y_{uuv}, y_{uvv}, y_{vvv}, and substituting for y_{uu}, y_{uv} their values from (16), we obtain

$$\left.\begin{aligned} y_{uuu} = \alpha^{(1)} y_{vv} + \beta^{(1)} y_u + \gamma^{(1)} y_v + \delta^{(1)} y, \\ y_{uuv} = \alpha^{(2)} y_{vv} + \beta^{(2)} y_u + \gamma^{(2)} y_v + \delta^{(2)} y, \\ y_{uvv} = \alpha^{(3)} y_{vv} + \beta^{(3)} y_u + \gamma^{(3)} y_v + \delta^{(3)} y, \\ y_{vvv} = \alpha^{(4)} y_{vv} + \beta^{(4)} y_u + \gamma^{(4)} y_v + \delta^{(4)} y, \end{aligned}\right\} \qquad (20)$$

where

$$\left.\begin{aligned} &\alpha^{(1)} = a\,(b + c') + a_u, \quad \beta^{(1)} = b_u + a b'_v + d + b^2 + b'\,(c + a b'), \\ &\qquad \gamma^{(1)} = c_u + a c'_v + a d' + b c + c'\,(c + a b'), \\ &\qquad \delta^{(1)} = d_u + a d'_v + b d + d'\,(c + a b'), \\ &\alpha^{(2)} = a b', \quad \beta^{(2)} = b'\,(b + c') + b'_u + d', \quad \gamma^{(2)} = b' c + c'^2 + c'_u, \\ &\qquad \delta^{(2)} = b' d + c' d' + d'_u, \\ &\alpha^{(3)} = c', \quad \beta^{(3)} = b'^2 + b'_v, \quad \gamma^{(3)} = b' c' + c'_v + d', \quad \delta^{(3)} = b' d' + d'_v, \\ &\alpha^{(4)} = \frac{1}{a}\,(a b' - c - a_v), \quad \beta^{(4)} = \frac{1}{a}\,(b' c' + b'_u - b_v + d'), \\ &\qquad \gamma^{(4)} = \frac{1}{a}\,[b' c + c'\,(c' - b) + c'_u - c_v - d], \\ &\qquad \delta^{(4)} = \frac{1}{a}\,[b' d + d'\,(c' - b) + d'_u - d'_v]. \end{aligned}\right\} \qquad (21)$$

In § 1, we saw that there could be just one integrability condition of the form

$$\alpha y_{vv} + \beta y_u + \gamma y_v + \delta y = 0, \tag{2 bis}$$

which gives four equations, $\alpha = \beta = \gamma = \delta = 0$. We may obtain these equations as follows. We must have

$$\frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \quad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \quad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}. \tag{22}$$

The second of these is satisfied identically by equations (20). Each of the other two gives a condition of the form (2), but the two sets of equations $\alpha = \beta = \gamma = \delta = 0$ found from them can not be independent, as may be verified by actual calculation. We need therefore use only one of equations (22), say the last. We obtain

$$\begin{aligned}\frac{\partial y_{uvv}}{\partial v} = {} & (\alpha^{(3)}\alpha^{(4)} + \gamma^{(3)} + \alpha^{(3)}_v)\, y_{vv} + (\alpha^{(3)}\beta^{(4)} + b'\beta^{(3)} + \beta^{(3)}_v)\, y_u \\ & + (\alpha^{(3)}\gamma^{(4)} + c'\beta^{(3)} + \gamma^{(3)}_v + \delta^{(3)})\, y_v + (\alpha^{(3)}\delta^{(4)} + d'\beta^{(3)} + \delta^{(3)}_v)\, y, \\ \frac{\partial y_{vvv}}{\partial u} = {} & (\alpha^{(4)}\alpha^{(3)} + a\beta^{(4)} + \alpha^{(4)}_u)\, y_{vv} + (\alpha^{(4)}\beta^{(3)} + b\beta^{(4)} + b'\gamma^{(4)} + \beta^{(4)}_u + \delta^{(4)})\, y_u \\ & + (\alpha^{(4)}\gamma^{(3)} + c\beta^{(4)} + c'\gamma^{(4)} + \gamma^{(4)}_u)\, y_v + (\alpha^{(4)}\delta^{(3)} + d\beta^{(4)} + d'\gamma^{(4)} + \delta^{(4)}_u)\, y.\end{aligned}$$

The two expressions on the right being identically equal, we find as the conditions of complete integrability:

$$\left.\begin{aligned} \gamma^{(3)} + \alpha^{(3)}_v &= a\beta^{(4)} + \alpha^{(4)}_u, \\ (\alpha^{(3)} - b)\,\beta^{(4)} + b'\beta^{(3)} + \beta^{(3)}_v &= \alpha^{(4)}\beta^{(3)} + b'\gamma^{(4)} + \beta^{(4)}_u + \delta^{(4)}, \\ c'\beta^{(3)} + \gamma^{(3)}_v + \delta^{(3)} &= \alpha^{(4)}\gamma^{(3)} + c\beta^{(4)} + \gamma^{(4)}_u, \\ \alpha^{(3)}\delta^{(4)} + d'\beta^{(3)} + \delta^{(3)}_v &= \alpha^{(4)}\delta^{(3)} + d\beta^{(4)} + d'\gamma^{(4)} + \delta^{(4)}_u. \end{aligned}\right\} \tag{23}$$

Use may be made of (21) to express these in terms of a, b, c, etc.; in fact, in the third of equations (23) we have already used the relation $\alpha^{(3)} = c'$. The first of equations (23) is found without difficulty to reduce to the equation

$$b_v + 2c'_v = \alpha^{(4)}_u + b'_u,$$

i. e., to

$$\frac{\partial}{\partial v}(b + 2c') = \frac{\partial}{\partial u}\Big(\frac{2ab' - c - a_v}{a}\Big).$$

We may therefore find by a quadrature a function p, given by

$$p_u = b + 2c', \quad p_v = \frac{2ab' - c - a_v}{a}. \tag{24}$$

§ 4. *Canonical Form of the System. The Seminvariants and Semicovariants.*

Let us carry out the transformation (8),

$$y = \lambda\,\bar{y},$$

on the system in its intermediate form. We have

$$\left.\begin{aligned} y_u &= \lambda\bar{y}_u+\lambda_u\bar{y}, \quad y_v = \lambda\bar{y}_v+\lambda_v\bar{y}, \\ y_{uu} &= \lambda\bar{y}_{uu}+2\lambda_u\bar{y}_u+\lambda_{uu}\bar{y}, \\ y_{uv} &= \lambda\bar{y}_{uv}+\lambda_v\bar{y}_u+\lambda_u\bar{y}_v+\lambda_{uv}\bar{y}, \\ y_{vv} &= \lambda\bar{y}_{vv}+2\lambda_v\bar{y}_v+\lambda_{vv}\bar{y}. \end{aligned}\right\} \tag{25}$$

Substituting in (16), we obtain a system of the same form in $\bar{y}$. Denoting its coefficients by $\bar{a}$, $\bar{b}$, etc., we find

$$\left.\begin{aligned} &\bar{a}=a, \quad \bar{b}=b-\frac{2\lambda_u}{\lambda}, \quad \bar{c}=c+\frac{2a\lambda_v}{\lambda}, \\ &\bar{d}=d+b\frac{\lambda_u}{\lambda}+c\frac{\lambda_v}{\lambda}-\frac{\lambda_{uu}}{\lambda}+a\frac{\lambda_{vv}}{\lambda}, \\ &\bar{a}'=0, \quad \bar{b}'=b'-\frac{\lambda_v}{\lambda}, \quad \bar{c}'=c'-\frac{\lambda_u}{\lambda}, \\ &\bar{d}'=d'+b'\frac{\lambda_u}{\lambda}+c'\frac{\lambda_v}{\lambda}-\frac{\lambda_{uv}}{\lambda}. \end{aligned}\right\} \tag{26}$$

The quantities p_u and p_v given by (24) are for the new system

$$\left.\begin{aligned} \bar{p}_u &= \bar{b}+2\bar{c}' = b+2c'-4\frac{\lambda_u}{\lambda}, \\ \bar{p}_v &= \frac{2\bar{a}\bar{b}'-\bar{c}-\bar{a}_v}{\bar{a}} = \frac{2ab'-c-a_v}{a}-4\frac{\lambda_v}{\lambda}. \end{aligned}\right\} \tag{27}$$

We may therefore make them vanish by choosing λ to satisfy the equations

$$p_u-4\frac{\lambda_u}{\lambda}=0, \quad p_v-4\frac{\lambda_v}{\lambda}=0,$$

i. e.,

$$\lambda = e^{p/4}. \tag{28}$$

Putting this value of λ into equations (26), we obtain a new set of coefficients, which we denote by capital letters:

$$\left.\begin{aligned} &A=a, \quad B=b-\frac{1}{2}p_u, \quad C=c+\frac{a}{2}p_v, \\ &D=d+\frac{1}{4}bp_u+\frac{1}{4}cp_v-\frac{1}{4}p_{uu}+\frac{1}{4}ap_{vv}-\frac{1}{16}p_u^2+\frac{1}{16}ap_v^2, \\ &B'=b'-\frac{1}{4}p_v, \quad C'=c'-\frac{1}{4}p_u, \\ &D'=d'+\frac{1}{4}b'p_u+\frac{1}{4}c'p_v-\frac{1}{4}p_{uv}-\frac{1}{16}p_up_v. \end{aligned}\right\} \tag{29}$$

The system of differential equations, of which these are the coefficients, will be called the *canonical form* of system (4). The canonical form is unique for any given system of the form (4), being completely characterized by the conditions

$$B+2C'=0, \quad 2AB'-C-A_v=0. \tag{30}$$

It is easily verified that, except for a factor, each of the coefficients (29) of

the canonical form remains unchanged under the transformation (8). We call a *seminvariant* any function of the coefficients of (16) and their derivatives which remains thus unchanged under the transformation (8). If such a function contain also y and its derivatives, we call it a *semi-covariant*. The seven quantities (29) are, then, seminvariants, as are also their derivatives for any order. But of the seven, at most five are independent in virtue of equations (30). We may reject two from our system, say B and C; we call the remaining five, viz., A, D, B', C', D', the *fundamental seminvariants*. The reason for the name is that *any seminvariant is a function of the five fundamental seminvariants and their derivatives.* In fact, a seminvariant must be unchanged by any transformation of the form (8), in particular by the transformation which puts the system of differential equations into the canonical form, and hence transforms the seminvariant into a function of the coefficients of the canonical form.

We now calculate the semi-covariants. We need seek only those containing y, y_u, y_v, y_{vv}, since all the derivatives of y are expressible in terms of these quantities. It is readily verified that

$$y, \quad \rho=y_u-c'y, \quad \sigma=y_v-b'y, \quad \tau=y_{vv}-2b'y_v+(b'^2-b'_v)y=\sigma_v-b'\sigma \tag{31}$$

satisfy the equations

$$\lambda\bar{y}=y, \quad \lambda\bar{\rho}=\rho, \quad \lambda\bar{\sigma}=\sigma, \quad \lambda\bar{\tau}=\tau, \tag{32}$$

and are therefore *relative* semi-covariants.

§ 5. *The Invariants.*

We found in § 2 that the most general transformation of the independent variables which leaves system (16) in the intermediate form, is of the form

$$\bar{u}=\phi(u), \quad \bar{v}=\psi(v), \tag{17}$$

where ϕ and ψ are any functions of their arguments. Let us apply this transformation to the system (16). We have

$$y_u=\bar{y}_u\phi_u, \quad y_v=\bar{y}_v\psi_v, \quad y_{uu}=\bar{y}_{uu}\phi_u^2+\bar{y}_u\phi_{uu},$$
$$y_{uv}=\bar{y}_{uv}\phi_u\psi_v, \quad y_{vv}=\bar{y}_{vv}\psi_v^2+\bar{y}_v\psi_{vv},$$

where $\bar{y}_u \equiv \partial y/\partial\bar{u}$, etc. Substituting in (16), we obtain for the coefficients of the new system

$$\left.\begin{aligned} \bar{a}=\frac{\psi_v^2}{\phi_u^2}a, \quad \bar{b}=\frac{1}{\phi_u}(b-\xi), \quad \bar{c}=\frac{\psi_v}{\phi_u^2}(c+a\eta), \quad \bar{d}=\frac{1}{\phi_u^2}d, \\ \bar{b}'=\frac{1}{\psi_v}b', \quad \bar{c}'=\frac{1}{\phi_u}c', \quad \bar{d}'=\frac{1}{\phi_u\psi_v}d', \end{aligned}\right\} \tag{33}$$

where

$$\xi=\phi_{uu}/\phi_u, \quad \eta=\psi_{vv}/\psi_v.$$

We have also

$$\left.\begin{aligned}\bar{p}_u=\frac{1}{\phi_u}(p_u-\xi),\quad \bar{p}_v=\frac{1}{\psi_v}(p_v-3\eta),\\ \bar{p}_{uu}=\frac{1}{\phi_u^2}(p_{uu}-\xi p_u+\xi^2-\xi_u),\quad \bar{p}_{uv}=\frac{1}{\phi_u\psi_v}p_{uv},\\ \bar{p}_{vv}=\frac{1}{\psi_v^2}(p_{vv}-\eta p_v+3\eta^2-3\eta_v).\end{aligned}\right\}\tag{34}$$

Using (33) and (34), we find that the fundamental seminvariants A, D, B', C', D' are transformed into

$$\left.\begin{aligned}\bar{A}=\frac{\psi_v^2}{\phi_u^2}A,\quad \bar{B}'=\frac{1}{\psi_v}(B'+\frac{3}{4}\eta),\quad \bar{C}'=\frac{1}{\phi_u}(C'+\frac{1}{4}\xi),\\ \bar{D}'=\frac{1}{\phi_u\psi_v}(D'-\frac{1}{4}B'\xi-\frac{3}{4}C'\eta-\frac{3}{16}\xi\eta),\\ \bar{D}=\frac{1}{\phi_u^2}(D-\frac{1}{4}B\xi-\frac{3}{4}C\eta-\frac{1}{16}\xi^2+\frac{9}{16}A\eta^2+\frac{1}{4}\xi_u-\frac{3}{4}A\eta_v).\end{aligned}\right\}\tag{35}$$

For the sake of symmetry, we have retained in the expression for $\bar{D}$ the quantities B and C, which may be expressed in terms of the fundamental seminvariants by means of (30). It will be convenient to have the transformed expressions for these quantities; they are

$$\bar{B}=\frac{1}{\phi_u}(B-\frac{1}{2}\xi),\quad \bar{C}=\frac{\psi_v}{\phi_u^2}(C-\frac{1}{2}A\eta).\tag{36}$$

We call *invariants* and *covariants* those seminvariants and semi-covariants which remain unchanged by a transformation of the form (17). By noting that

$$\left.\begin{aligned}\bar{A}_u=\frac{\psi_v^2}{\phi_u^3}(A_u-2A\xi),\quad \bar{A}_v=\frac{\psi_v}{\phi_u^2}(A_v+2A\eta),\\ \bar{B}'_v=\frac{1}{\psi_v^2}(B'_v-B'\eta-\frac{3}{4}\eta^2+\frac{3}{4}\eta_v),\quad \bar{C}'_u=\frac{1}{\phi_u^2}(C'_u-C'\xi-\frac{1}{4}\xi^2+\frac{1}{4}\xi_u),\end{aligned}\right\}\tag{37}$$

we may verify the invariance of

$$\left.\begin{aligned}\mathfrak{A}=A,\quad \mathfrak{B}'=B'-\frac{3}{8}\frac{A_v}{A},\quad \mathfrak{C}'=C'+\frac{1}{8}\frac{A_u}{A},\\ \mathfrak{D}'=D'+B'C',\quad \mathfrak{D}=D-(B'A_v-AB'_v)-C'_u+3(AB'^2-C'^2).\end{aligned}\right\}\tag{38}$$

These are *relative* invariants, satisfying the equations

$$\bar{\mathfrak{A}}=\frac{\psi_v^2}{\phi_u^2}\mathfrak{A},\quad \bar{\mathfrak{B}}'=\frac{1}{\psi_v}\mathfrak{B}',\quad \bar{\mathfrak{C}}'=\frac{1}{\phi_u}\mathfrak{C}',\quad \bar{\mathfrak{D}}'=\frac{1}{\phi_u\psi_v}\mathfrak{D}',\quad \bar{\mathfrak{D}}=\frac{1}{\phi_u^2}\mathfrak{D}.\tag{39}$$

By means of (38), we may express the fundamental seminvariants A, D, B', C', D' entirely in terms of the invariants $\mathfrak{A}$, $\mathfrak{D}$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}'$ and their first derivatives. It follows that *every invariant is a function of the invariants* (38) *and their derivatives*, since an invariant is a function of the fundamental seminvariants and their derivatives.

If we suppose that $\mathfrak{B}'$ and $\mathfrak{C}'$ are not zero, we may form from the five relative invariants (38) the three *absolute* invariants $\mathfrak{A}\mathfrak{B}'^2/\mathfrak{C}'^2$, $\mathfrak{D}'/\mathfrak{B}'\mathfrak{C}'$, $\mathfrak{D}/\mathfrak{C}'^2$. Moreover, it is not difficult to see that any relative invariant $I^{(l,m)}$ must be transformed by (17) into an expression of the form

$$\bar{I}^{(l,m)} = \phi_u^l \psi_v^m I^{(l,m)};$$

$\mathfrak{B}'$ and $\mathfrak{C}'$ may then be used to form from $I^{(l,m)}$ an absolute invariant $\mathfrak{B}'^m \mathfrak{C}'^l I^{(l,m)}$, or again to form two invariants which satisfy the relations

$$\bar{\mathfrak{B}}'^m \bar{I}^{(l,m)} = \phi_u^l \mathfrak{B}'^m I^{(l,m)}, \quad \bar{\mathfrak{C}}'^l \bar{I}^{(l,m)} = \psi_v^m \mathfrak{C}' \bar{I}^{(l,m)}.$$

We call an invariant a ϕ-invariant, or a ψ-invariant, if its transform is multiplied by a power of ϕ_u alone, or by a power of ψ_u alone. Thus, from (39) we have the

$$\phi\text{-invariants } \mathfrak{C}', \mathfrak{D}, \mathfrak{B} = \mathfrak{A}\mathfrak{B}'^2, \quad \psi\text{-invariants } \mathfrak{B}', \mathfrak{C} = \mathfrak{D}'/\mathfrak{C}'. \tag{40}$$

Consider the operators

$$U = \frac{1}{\mathfrak{C}'}\frac{\partial}{\partial u}, \quad V = \frac{1}{\mathfrak{B}'}\frac{\partial}{\partial v}.$$

Let Φ be a ϕ-invariant, Ψ a ψ-invariant. Then $U(\Psi)$ is a new ψ-invariant, and $V(\Phi)$ a new ϕ-invariant.

Again, from two ϕ-invariants $\Phi^{(l)}$ and $\Phi^{(m)}$, for which $\bar{\Phi}^{(l)} = \phi_u^l \Phi^{(l)}$, $\bar{\Phi}^{(m)} = \phi_u^m \Phi^{(m)}$, we may form a new ϕ-invariant, its Wronskian with respect to u:

$$(\Phi^{(m)}, \Phi_u^{(l)}) \equiv l\,\Phi^{(l)}\Phi_u^{(m)} - m\,\Phi^{(m)}\Phi_u^{(l)}.$$

Starting with the invariants (40)—the three ϕ-invariants $\mathfrak{C}'$, $\mathfrak{D}$, $\mathfrak{B}$, and the two ψ-invariants $\mathfrak{B}'$, $\mathfrak{C}$—we may construct an infinite number of ϕ- and ψ-invariants by combining the Wronskian process with the U- and V-processes. It may be shown, by an argument parallel to that employed by Wilczynski in his first memoir on curved surfaces,* that *by means of the Wronskian process and the U- and V-processes all invariants may be deduced from the five fundamental invariants $\mathfrak{C}'$, $\mathfrak{D}$, $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{C}$. Of the complete system, there are $(n+1)(5n+6)/2$ absolute, or two more relative, invariants which contain derivatives of the fundamental invariants of orders up to and including n, and which are independent of each other and the five fundamental invariants.*

We must, however, note a restriction on the independence of the complete system of invariants. In the presence of the integrability conditions (23), they are redundant. In fact, equations (23) must be invariant under the transformations (8) and (17), and therefore expressible in terms of invariants. The first of (23) has already been used to give the relations (30); it may be verified without difficulty that the other three integrability conditions give three relations connecting the five fundamental invariants (40) and their first and second derivatives.

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), pp. 250–255.

If we have given the fundamental invariants (40) (or what is the same thing, the invariants (38)), satisfying the integrability conditions (23), we may always find the five fundamental seminvariants and therefore the seven coefficients of the canonical form of the system of differential equations. A transformation of the form

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v) \tag{17}$$

will put the system into the intermediate form; and more generally still, a transformation

$$\bar{u} = U(u, v), \qquad \bar{v} = V(v) \tag{9}$$

will throw the system into the most general form (4). *Thus the system of curves C_u is completely determined, except for a projective transformation, by the fundamental invariants* (40), *or at any rate by the invariants* (38).

Geometrically it is evident that the invariants of the conjugate net are invariants of the family of curves C_u. It may be expected, therefore, that the invariants, as we have calculated them for the intermediate form of the system of differential equations, are in some way or other expressible in terms of the coefficients of the original system (4). It turns out, in fact, that to obtain these expressions it is not necessary to integrate the differential equation (14), although such integration is necessary in obtaining the coefficients of the intermediate form. Equations (19) give the coefficients of the intermediate form in terms of the coefficients of system (4), and in these expressions appear the quantities U_u and U_{uu}, the values of which may be found only on integrating (14). But if the values for the quantities $\bar{a}$, $\bar{b}$, etc., be substituted in (38), and if use be made of equations (14) and (18) to express all derivatives of U in terms of U_u and U_{uu}, it will be found that U_{uu} disappears, and that the invariants have the values

$$\mathfrak{A} = \frac{1}{U_u^2}\mathsf{A}, \quad \mathfrak{B}' = \mathsf{B}', \quad \mathfrak{C}' = \frac{1}{U_u}\mathsf{C}', \quad \mathfrak{D}' = \frac{1}{U_u}\mathsf{D}', \quad \mathfrak{D} = \frac{1}{U_u^2}\mathsf{D}, \tag{41}$$

where $\mathsf{A}, \mathsf{B}', \mathsf{C}', \mathsf{D}', \mathsf{D}$ are expressions in a, b, etc. (the coefficients of system (4)) and their derivatives, which expressions take the form of the quantities $\mathfrak{A}, \mathfrak{B}', \mathfrak{C}', \mathfrak{D}', \mathfrak{D}$ if we put $a' = 0$. We get therefore for the ϕ- and ψ-invariants (40):

$$\phi\text{-invariants}, \quad \mathfrak{C}' = \frac{1}{U_u}\mathsf{C}', \quad \mathfrak{D} = \frac{1}{U_u^2}\mathsf{D}, \quad \mathfrak{B} = \frac{1}{U_u^2}\mathsf{B},$$

$$\psi\text{-invariants}, \quad \mathfrak{B}' = \mathsf{B}', \quad \mathfrak{C} = \mathsf{C},$$

where $\mathsf{B} = \mathsf{A}/\mathsf{B}'^2$, $\mathsf{C} = \mathsf{D}'/\mathsf{C}'$. The fundamental invariants are therefore expressible explicitly in terms of the coefficients of system (4) and their derivatives. Only for the the three ϕ-invariants is there an extraneous factor, some power of U_u.

We may prove by induction that the invariants derived from the fundamental invariants by means of the U-, V- and Wronskian processes are similarly expressible in terms of the coefficients of (4). For the transformation

$$\bar{u} = U(u, v), \quad \bar{v} = v \tag{10}$$

we have (using (14))

$$\frac{\partial}{\partial \bar{u}} = \frac{1}{U_u}\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial \bar{v}} = \frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u},$$

where $\bar{u}$ and $\bar{v}$ are the variables in the intermediate form, and u, v the variables in (4). Consequently, the operators U and V become

$$\mathbf{U} = \frac{1}{\mathbf{C}'}\frac{\partial}{\partial u}, \quad \mathbf{V} = \frac{1}{\mathbf{B}'}\left(\frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u}\right).$$

Assume that every ϕ-invariant $\mathfrak{M}$ up to a certain order is such that on transformation by (19) it becomes

$$\mathfrak{M} = U_u^l \mathbf{M}.$$

We then have

$$V(\mathfrak{M}) = \mathbf{V}(\mathbf{M}) = \frac{l\,U_u^{l-1}}{a\mathbf{B}'}\mathbf{M}(aU_{uv} - a'U_{uu}) + \frac{U_u^l}{\mathbf{B}'}\left(\frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u}\right)\mathbf{M}.$$

But by (18) we have

$$aU_{uv} - a'U_{uu} = a'_u U_u - a_u U_v = \frac{U_u}{a}(aa'_u - a'a_u),$$

so that

$$\mathbf{V}(\mathbf{M}) = \frac{U_u^l}{\mathbf{B}'}\left[\frac{l(aa'_u - a'a_u)}{a^2}\mathbf{M} + \left(\frac{\partial}{\partial v} - \frac{a'}{a}\frac{\partial}{\partial u}\right)\mathbf{M}\right].$$

The new invariant is therefore expressible in the required way, and is multiplied again by the factor U_u^l.

Assume further that up to a certain order every ψ-invariant $\mathfrak{N}$ is transformed by (19) in such a way that

$$\mathfrak{N} = \mathbf{N},$$

there being no extraneous factor, and $\mathbf{N}$ being the expression in the coefficients of (4). Then

$$U(\mathfrak{N}) = \mathbf{U}(\mathbf{N}) = \frac{1}{\mathbf{C}'}\mathbf{N}_u,$$

and is also expressible in the required way.

Now, the hypotheses as to the nature of the transforms of $\mathfrak{M}$ and $\mathfrak{N}$ are satisfied by the transforms (41) of the five fundamental invariants. The induction therefore follows, as far as the U- and V-processes are concerned. The proof as regards the Wronskian process is essentially the same. We may therefore state the important theorem:

All the invariants of the conjugate net of curves C_u and C_v are expressible explicitly in terms of the coefficients of system (4), *the integration of the differential equation* (14) *not being necessary even though such integration is required to put system* (4) *into the intermediate form.*

§ 6. *The Covariants and their Geometric Interpretation. Laplace Transformations.*

The semi-covariants (31) are transformed by (17) into

$$\bar{y} = y, \quad \bar{\rho} = \frac{1}{\phi_u}\rho, \quad \bar{\sigma} = \frac{1}{\psi_v}\sigma, \quad \bar{\tau} = \frac{1}{\psi_v^2}\ \tau - \eta\sigma),$$

where $\eta = \psi_{vv}/\psi_v$. The quantities y, ρ, σ are therefore covariants; in giving their geometric interpretation we shall find a fourth covariant to replace the quantity τ, which is not itself a covariant.

Analytically, the covariants

$$\rho = y_u - c' y, \quad \sigma = y_v - b' y$$

are transformations introduced by Laplace* in his study of the differential equation

$$y_{uv} = b' y_u + c' y_v + d' y.$$

Darboux† has given a very elegant geometric interpretation of the Laplace transformations, which we now outline.

If the four coordinates

$$y^{(k)} = f^{(k)}(u, v), \qquad (k = 1, 2, 3, 4), \tag{1 bis}$$

of a point y (or P_y) be substituted successively in the expressions for ρ and σ, we obtain the coordinates of two points, P_ρ and P_σ. Thus, for given values of u and v we get a point y and two associated points ρ and σ. To the surface S_0 given by equations (1 *bis*) correspond therefore two surfaces S_{-1} and S_1 given by the covariants ρ and σ. For $v =$ const., we get a curve C_u on S_0, and corresponding curves on S_{-1} and S_1.

We suppose that $y^{(k)} = f^{(k)}(u, v)$ are a fundamental system of solutions of the system of differential equations in the intermediate form,‡ so that the curves

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* "Recherches sur le Calcul intégral aux différences partielles," *Œuvres de Laplace*, t. IX, pp. 29 *et seq.*

† "Théorie Générale des Surfaces," t. II, livre IV, chap. I, II.

‡ Darboux does not consider a system of two differential equations, but only a single equation of the Laplace type:

$$y_{uv} = b' y_u + c' y_v + d' y.$$

For the geometric interpretation of the covariants, this single equation is sufficient; but the results have a deeper significance if with it we associate another equation to form the system (16). In particular, a projective theory is not possible with a single differential equation; but more than this, the introduction of a second equation not only makes the theory projective, but enables us to study the single system of curves C_u apart from the conjugate net. The force of this last remark will appear in the sequel.

u=const. and v=const. form a conjugate net on the surface S_0. The tangents to the curves v=const. form a congruence Γ_{-1}, one sheet of whose focal surface is S_0. The other sheet will be found by determining the edges of regression of the developables generated by the lines of the congruence which meet fixed curves of the family u=const. conjugate to the family v=const. Any point on the tangent at a point P_y to the curve v=const. which passes through P_y will have coordinates

$$Y^{(k)} = y_u^{(k)} + \lambda\, y^{(k)}, \qquad (k = 1, 2, 3, 4).$$

Suppose λ to be any function of u, v, and let P_y trace a curve u=const. Then the point P_Y traces a curve, and a point on the tangent to this curve at P_Y is given by

$$\begin{aligned}\frac{\partial Y^{(k)}}{\partial v} &= y_{uv}^{(k)} + \lambda\, y_v^{(k)} + \lambda_v\, y^{(k)}\\ &= b'\, y_u^{(k)} + (\lambda + c')\, y_v^{(k)} + (\lambda_v + d')\, y^{(k)}.\end{aligned}$$

If $\lambda = -c'$, the point Y becomes the point ρ, and we have

$$\frac{\partial \rho}{\partial v} = b'\, y_u + (d' - c'_v)\, y.$$

But then the point ρ_v lies on the line of the congruence Γ_{-1}, and the point ρ traces the required edge of regression. Therefore, *the covariant*

$$\rho = y_u - c'\, y$$

gives the second sheet S_{-1} of the congruence Γ_{-1} of tangents to the curves v=const. on the surface S_0.

Similarly, *the congruence Γ_1 of tangents to the curves u=const. on S_0 has as the second sheet S_1 of its focal surface the points given by the covariant*

$$\sigma = y_v - b'\, y.$$

We call ρ and σ respectively the *minus first* and *first Laplace transforms* of y, and correspondingly the surfaces S_{-1} and S_1 the *minus first* and *first Laplace transforms* of S_0. The first Laplace transform of S_{-1} is S_0, and its minus first Laplace transform is a surface S_{-2}, which is given by a covariant ρ_{-1}. We easily find the expression for ρ_{-1}. We have

$$\begin{aligned}\rho_v &= b'\, y_u + (d' - c'_v)\, y\\ &= b'\, \rho + (d' + b'\, c' - c'_v)\, y,\end{aligned}$$

The parenthesis is an invariant; we write

$$K = d' + b'\, c' - c'_v = D' + B'\, C' - C'_v, \tag{42}$$

and get

$$\left.\begin{aligned}\rho_v &= b'\, \rho + K\, y,\\ \rho_{uv} &= b'\, \rho_u + b'_u\, \rho + K\, y_u + K_u\, y,\end{aligned}\right\} \tag{43}$$

or, using (43),

$$\rho_{uv}=b'\rho_u+\left(c'+\frac{K_u}{K}\right)\rho_v+\left[b'_u+K-b'\left(c'+\frac{K_u}{K}\right)\right]\rho. \tag{44}$$

Consequently, the minus first Laplace transform of ρ is

$$\rho_{-1}=\rho_u-\left(c'+\frac{K_u}{K}\right)\rho. \tag{45}$$

It is of course a covariant of the original conjugate net on S_0, as may be verified by calculation.

The first Laplace transform of ρ is $\rho_v-b'\rho$, which by (43) is simply Ky, so that as stated above the first Laplace transform of S_{-1} is S_0.

Similarly, the minus first Laplace transform of S_1 is S_0, and its first Laplace transform is a new surface S_2 given by the covariant

$$\sigma_1=\sigma_v-\left(b'+\frac{H_v}{H}\right)\sigma, \tag{46}$$

where

$$H=d'+b'c'-b'_u=D'+B'C'-B'_u \tag{47}$$

is an invariant.

Equation (44), and the analogous equation for σ,

$$\sigma_{uv}=\left(b'+\frac{H_v}{H}\right)\sigma_u+c'\sigma_v+\left[c'_v+H-c'\left(b'+\frac{H_v}{H}\right)\right]\sigma, \tag{48}$$

show that the curves $u=$const. and $v=$const. are conjugate on the surfaces S_{-2} and S_2. By repeating the Laplace transformations in the two directions, we obtain an infinite sequence of surfaces $\ldots, S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2, \ldots$; and on each the families of curves $u=$const. and $v=$const. will be conjugate. For the surface S_k, there will be a pair of congruences Γ_{k+1}, Γ_{k-1}, formed respectively by the tangents to the curves $u=$const., $v=$const.

The above is Darboux's very elegant geometric interpretation of the Laplace transformations of the partial differential equation

$$y_{uv}=b'y_u+c'y_v+d'y.$$

But for our projective theory we have added another partial differential equation of the second order to form the completely integrable system (16). We recall that system (16) is the intermediate form of the system of differential equations (4), and requires for its determination the integration of a partial differential equation of the first order, viz., equation (14). This, as we know, is equivalent to the determination of the family of curves conjugate to the family of curves C_u, or $v=$const. This is necessary for the application of Darboux's theory, since the surface is there referred to a conjugate net of parameter curves. However, it is evident geometrically that the Laplace

transforms of the surface S_0 are determined by the single system of curves $v=\text{const}$. *By the association of a second differential equation with Darboux's single equation, the Laplace transforms of a surface defined by a one-parameter family of curves may be found without the determination of the conjugate family of curves.*

The proof is very simple. Let us return to the notation of § 2, at the end of which we denoted the coefficients of the intermediate form by barred letters. In this notation the covariants ρ and σ are

$$\rho = \bar{y}_u - \bar{c}'\,\bar{y}, \quad \sigma = \bar{y}_v - \bar{b}'\,\bar{y}.$$

But we may express these quantities in terms of the variables and coefficients of system (4), which we denote by unbarred letters. From (11), we have

$$\bar{y}_u = \frac{1}{U_u} y_u, \quad \bar{y}_v = y_v - \frac{a'}{a} y_u,$$

in the latter of which we have made use of (14). From (19) we may take the values for $\bar{b}'$ and $\bar{c}'$, and thus obtain

$$\rho = \frac{1}{U_u}\left[y_u - \frac{a c' - a' c}{a} y\right], \tag{49}$$

$$\sigma = y_v - \frac{a'}{a} y_u - \frac{a(ab'-a'b) + a'(ac'-a'c) + a' a_u - a a'_u}{a^2} y. \tag{50}$$

The covariants ρ and σ are therefore expressed in terms of the variables and coefficients of system (4): σ entirely so, and ρ except for the factor U_u which can not be determined without the integration of equation (14). But geometrically only the ratios of the homogeneous coordinates of a point are significant; hence the first and minus first Laplace transforms of S_0 are indeed determined without the integration of any differential equation. We may say, in the language of Lie, that *the Laplace transforms of a one-parameter family of space curves given by equations* (1) *may be found from these equations by "performable operations" alone*; and not only are these operations "performable" in the very general sense of Lie, but also in the sense that *all of the algebraic processes required to obtain the explicit expressions may actually be carried out in practice.*

We may now set up our complete system of covariants. Since all derivatives of y are expressible in terms of y, y_u, y_v, y_{vv}, we need seek only covariants which contain these four quantities. We have already found these covariants; they are:

$$\left.\begin{array}{c} y, \quad \rho = y_u - c' y, \quad \sigma = y_v - b' y, \\ \sigma_1 = \sigma_v - \left(b' + \dfrac{H_v}{H}\right)\sigma = y_{vv} - \left(2b' + \dfrac{H_v}{H}\right) y_v + b'\left(b' + \dfrac{H_v}{H} - \dfrac{b'_v}{b'}\right) y. \end{array}\right\} \tag{51}$$

We may take these as our fundamental system of covariants. *Every covariant is a function of these fundamental covariants and of invariants.*

It is easily verified that σ_1 is, like ρ and σ, expressible explicitly in terms of the coefficients and variables of system (4). We recall that the complete system of invariants are also expressible in this way. It is evident geometrically that every invariant and covariant of a one-parameter family of curves is an invariant or covariant of the conjugate net of which this family is a part, and conversely. Also, analytically, every invariant and covariant of system (4) under the transformations of the group

$$y = \lambda\,(u, v)\,\bar{y}, \quad \bar{u} = U\,(u, v), \quad \bar{v} = V\,(v)$$

is certainly an invariant or covariant of system (16) under the transformations of the smaller group

$$y = \lambda\,(u, v)\,\bar{y}, \quad \bar{u} = U\,(u), \quad \bar{v} = V\,(v).$$

From these considerations we may conclude that *in the projective study of a one-parameter family of space curves there is no loss of generality, from an analytic point of view,* in considering instead the conjugate net of which the given system of curves is a component family.*

Since the quantities ρ and σ define two surfaces referred to conjugate nets as parameter curves, each of them must satisfy a system of differential equations of form (16). Equations (44) and (48) are respectively the second equations of the systems for ρ and σ. We shall merely indicate how the first equation of each system is found. We have

$$\left.\begin{aligned} \rho &= y_u - c'y, & \rho_u &= a y_{vv} + (b-c')y_u + c y_v + (d-c'_u)y, \\ \rho_v &= b'y_u + (d'-c'_v)y, & \rho_{vv} &= (b'^2+b'_v)y_u + K y_v + (d'_v+b'd'-c'_{vv})y, \end{aligned}\right\} \quad (52)$$

and an expression for ρ_{uu} linear in y_{vv}, y_u, y_v, y. From this last and equations (52) we may eliminate the quantities y_{vv}, y_u, y_v, y, and obtain the required equation for ρ. This elimination will be impossible, however, if and only if the four equations (52) are linearly dependent in y_{vv}, y_u, y_v, y, in which case there will be a relation of the form

$$\alpha\rho_{vv} + \beta\rho_u + \gamma\rho_v + \delta\rho = 0,$$

excluded in § 1. The condition for this is the vanishing of the determinant

$$\begin{vmatrix} 0 & 1 & 0 & -c' \\ a & b-c' & c & d-c'_u \\ 0 & b' & 0 & d'-c'_v \\ 0 & b'^2+b'_v & K & d'_v+b'd'-c'_{vv} \end{vmatrix} = aK^2.$$

Since we have supposed $a \neq 0$, we need consider only the vanishing of K. We shall soon show that *if $K = 0$, the minus first Laplace transform of S_0 is degenerate.*

* Nor, obviously, from a geometric point of view.

For σ, we have

$$\left.\begin{array}{c}\sigma = y_v - b'y, \quad \sigma_u = c'y_v + (d' - b'_u)y, \quad \sigma_v = y_{vv} - b'y_v - b'_v y, \\ \sigma_{vv} = (\alpha^{(4)} - b')y_{vv} + \beta^{(4)}y_u + (\gamma^{(4)} - 2b'_v)y_v + (\delta^{(4)} - b'_{vv})y,\end{array}\right\} \tag{53}$$

in the last of which $\alpha^{(4)}$, $\beta^{(4)}$, $\gamma^{(4)}$, $\delta^{(4)}$ are the coefficients in the equation

$$y_{vvv} = \alpha^{(4)}y_{vv} + \beta^{(4)}y_u + \gamma^{(4)}y_v + \delta^{(4)}y, \tag{54}$$

and are given by (21). Between (53) and a similar expression for σ_{uu}, the quantities y_{vv}, y_u, y_v, y may in general be eliminated, and the result will be the required equation for σ. However, as in the above, the elimination is impossible if the determinant of the coefficients of y_{vv}, y_u, y_v, y in (53) vanishes. This determinant is

$$\begin{vmatrix} 0 & 0 & 1 & -b' \\ 0 & 0 & c' & d' - b'_u \\ 1 & 0 & -b' & -b'_v \\ \alpha^{(4)} - b' & \beta^{(4)} & \gamma^{(4)} - 2b'_v & \delta^{(4)} - b'_{vv} \end{vmatrix} = \beta^{(4)}H.$$

Obviously $\beta^{(4)}$ is an invariant; in fact, its value in (24) is easily reduced to

$$\beta^{(4)} = \frac{1}{A}\,(H + 2B'_u + 2C'_v). \tag{55}$$

Hence, *if either $\beta^{(4)} = (H + 2B'_u + 2C'_v)/A$ or $H = d' + b'c' - b'_u$ vanishes, the first Laplace transform of the surface S_0 is developable or degenerate.* We shall see that if $H = 0$, the first Laplace transform is not developable, but degenerate.

There is a certain dissymmetry in the statement of the last two theorems for ρ and σ, which may easily be removed. From (54), we see that if $\beta^{(4)} = 0$, we have

$$y_{vvv} = \alpha^{(4)}y_{vv} + \gamma^{(4)}y_v + \delta^{(4)}y;$$

in other words, *the vanishing of the invariant $\beta^{(4)} = (H + 2B'_u + 2C'_v)/A$ is the condition that the curves $u = const.$ be plane curves* (not of course lying in the same plane).

It is easy to see geometrically that if the curves u=const. be plane, the first Laplace transform is developable or degenerate. In fact, this transform is the developable (which may be degenerate) enveloped by the family of planes which cut out the curves u=const. on the surface S_0.

It should be noted that in Darboux's theory the case $\beta^{(4)} = 0$ is not exceptional, since the single equation (48) can always be set up if H is not zero. For the minus first Laplace transform, ρ, it is always possible to find both the equation in ρ_{uu} and the equation in ρ_{uv}, provided only that K does not vanish. But the curves v=const. may be plane, and the surface S_{-1} developable or degenerate. Then we should expect the coefficient of ρ_{vv} in the ρ_{uu}-equation to be zero. This is easily seen to be the case. We have

$$y_{uu} = a y_{vv} + b y_u + e y_v + d y,$$
$$y_{uuu} = \alpha^{(1)} y_{vv} + \beta^{(1)} y_u + \gamma^{(1)} y_v + \delta^{(1)} y,$$

where $\alpha^{(1)}$, $\beta^{(1)}$, $\gamma^{(1)}$, $\delta^{(1)}$ are given by (21). *The condition that the curves $v = const.$ be plane is therefore the vanishing of the determinant*

$$\begin{vmatrix} a & c \\ \alpha^{(1)} & \gamma^{(1)} \end{vmatrix} = A^2 [H + 3\mathfrak{B}'_u + \mathfrak{C}'_v], \tag{56}$$

which is of course an invariant.

Now,

$$\begin{aligned} \rho_{uu} &= y_{uuu} - c'_{uu} y - 2 c'_u y_u - c' y_{uu} \\ &= (\alpha^{(1)} - a c') y_{vv} + (\beta^{(1)} - 2 c'_u - b c') y_u \\ &\quad + (\gamma^{(1)} - c c') y_v + (\delta^{(1)} - c'_{uu} - d c') y, \end{aligned}$$

and if the determinant $a\gamma^{(1)} - c\alpha^{(1)} = 0$, we may eliminate, from this equation for ρ_{uu} and the equations for ρ, ρ_u, ρ_v given by (52), the quantities y_{vv}, y_u, y_v, y. The result is an equation of the form

$$\rho_{uu} = \beta \rho_u + \gamma \rho_v + \delta \rho,$$

which associated with equation (44) can define only a developable or degenerate surface.

We may therefore state the completed theorem, the last part of which is still to be proved: *If the curves $u = const.$ are plane, the first Laplace transform of S_0 is developable or degenerate. If the curves $v = const.$ are plane, the minus first Laplace transform is developable or degenerate. The only other cases of a similar nature are those in which the invariant H (or the invariant K) vanishes, in which event the first (or minus first) Laplace transform is degenerate.*

To prove the last part of this theorem, it is only necessary to note that for $K = 0$ we have $\rho_v = b'\rho$. This shows that only one point ρ corresponds to all the points of a curve $u = $const. on S_0, and that corresponding to the family of curves $u = $const. we have the points of a curve, say C_ρ. This is the degenerate surface S_1. In the same way we get a degenerate surface S_{-1} if $H = 0$. We may therefore describe these degenerate cases as follows in terms of the original one-parameter family of curves. *If the invariant H vanishes, the curves $C_u (v = const.)$ are the curves along which the surface S_0 is touched by a family of cones enveloping the surface and having their vertices on a curve in space. If K vanishes, the curves C_u are conjugate to a family of curves of contact of cones enveloping the surface and having their vertices on a curve in space.*

The invariants H and K, and the similar quantities for the different Laplace transforms, are important analytically in the study of the differential equation

$$y_{uv}=b'y_u+c'y_v+d',$$

as was first pointed out by Laplace. In fact, if either H or K vanishes, the differential equation is soluble by quadratures, and the same is true for the i-th Laplace transform (i positive or negative) of the differential equation, if either of the corresponding invariants H_i, K_i vanishes. This theory is well known through Darboux's researches, and need not be gone into here. It should be noted, however, that in our theory we deal with two partial differential equations, so that a number of new analytical problems arise in this connection. For instance, if one of the equations may be solved by quadratures, the solutions must still be further restricted so as to satisfy the other differential equation.*

§ 7. *The Surface Referred to its Asymptotic Lines.*

In the first three of his five memoirs on curved surfaces,† Wilczynski studied non-developable surfaces referred to their asymptotic curves. We shall now show how the results of his investigations may be made available for the theory of a one-parameter family of space curves.

The equations in the intermediate form (16) have S_0 as an integral surface. This surface will be left unchanged by any transformation of the form

$$\bar{u}=\phi\,(u,v),\qquad \bar{v}=\psi\,(u,v). \tag{57}$$

This transformation refers S_0 to a different net of parameter curves, and we seek the particular transformation which converts the conjugate net on S_0 into the net of asymptotic curves.

Let us carry out the transformation (57) on system (16). We obtain a system of two equations of the second order, with derivatives of y with respect to $\bar{u}$ and $\bar{v}$. Denote these derivatives by $\bar{y}_u$, $\bar{y}_v$, etc., and solve for the derivatives $\bar{y}_{uu}$ and $\bar{y}_{vv}$ in terms of $\bar{y}_{uv}$, $\bar{y}_u$, $\bar{y}_v$, $\bar{y}$. The result is the two equations

$$\left.\begin{aligned}\bar{y}_{uu}&=\alpha\,\bar{y}_{uv}+\beta\,\bar{y}_u+\gamma\,\bar{y}_v+\delta\,\bar{y},\\ \bar{y}_{vv}&=\alpha'\,\bar{y}_{uv}+\beta'\,\bar{y}_u+\gamma'\,\bar{y}_v+\delta'\,\bar{y},\end{aligned}\right\} \tag{58}$$

where

$$\left.\begin{aligned}\Delta\,\alpha\;&=-(\phi_u\psi_v-\phi_v\psi_u)\,(\psi_u^2+a\psi_v^2),\\ \Delta\,\alpha'&=(\phi_u\psi_v-\phi_v\psi_u)\,(\phi_u^2+a\phi_v^2),\\ \Delta\,\gamma\;&=-\psi_v(\psi_u\psi_{uu}+a\psi_v\psi_{uv})+\psi_u(\psi_u\psi_{uv}+a\psi_v\psi_{vv})\\ &\qquad+\psi_v[b\psi_u^2-c'(\psi_u^2-a\psi_v^2)]+\psi_u[c\psi_v^2-b'(\psi_u^2-a\psi_v^2)],\\ \Delta\,\beta'&=\phi_v(\phi_u\phi_{uu}+a\phi_v\phi_{uv})-\phi_u(\phi_u\phi_{uv}+a\phi_v\phi_{vv})\\ &\qquad-\phi_v[b\phi_u^2-c'(\phi_u^2-a\phi_v^2)]-\phi_u[c\phi_v^2-b'(\phi_u^2-a\phi_v^2)],\end{aligned}\right\} \tag{59}$$

* The same remark is made by Wilczynski in a similar connection, in his memoir "One-parameter Families and Nets of Plane Curves," *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 493.

† *Transactions of the American Mathematical Society*, Vols. VIII, IX, X (1907–09).

$$\Delta=(\phi_u\psi_v-\phi_v\psi_u)(\phi_u\psi_u+a\phi_v\psi_v), \tag{60}$$

and the remaining coefficients, whose values do not concern us, are fractions with the common denominator Δ.

We may make α and α' vanish, by taking for $\phi_u:\phi_v$ and $\psi_u:\psi_v$ the two roots of the quadratic

$$t^2+a=0,$$

or more definitely, by taking for ϕ and ψ solutions of the partial differential equations of the first order:

$$\left.\begin{aligned}\phi_u+\sqrt{-a}\,\phi_v=0,\\ \psi_u-\sqrt{-a}\,\psi_v=0.\end{aligned}\right\} \tag{61}$$

If this be done, we see from the resulting form of (58) that the curves $\bar{u}=$const. and $\bar{v}=$const. are asymptotic.*

By differentiation of $\phi_u^2+a\phi_v^2=0$, we have

$$\phi_u\phi_{uu}+a\phi_v\phi_{uv}=-\frac{a_u}{2}\phi_v^2,$$

$$\phi_u\phi_{uv}+a\phi_v\phi_{vv}=-\frac{a_v}{2}\phi_v^2,$$

with analogous equations for ψ. Using these and (61), we find for the quantities (59) and (60):

$$\Delta=-4a\sqrt{-a}\,\phi_v^2\psi_v^2,$$

which we now see is not zero, and

$$\left.\begin{aligned}\gamma&=-\frac{\psi_v}{\sqrt{-a}\,\phi_v^2}\left[c'-\frac{b+2c'}{4}+\frac{1}{8}\frac{a_u}{a}+\sqrt{-a}\left(b'-\frac{2ab'-c-a_v}{4a}-\frac{3}{8}\frac{a_v}{a}\right)\right],\\ \beta'&=\frac{\phi_v}{\sqrt{-a}\,\psi_v^2}\left[c'-\frac{b+2c'}{4}+\frac{1}{8}\frac{a_u}{a}-\sqrt{-a}\left(b'-\frac{2ab'-c-a_v}{4a}-\frac{3}{8}\frac{a_v}{a}\right)\right].\end{aligned}\right\} \tag{62}$$

Thus, except for the factors outside the square brackets, both γ and β' are expressible explicitly in terms of the coefficients of (16). The remaining coefficients of (58) will also be thus expressible, except that from the expressions for β and γ' one of the second derivatives of ϕ and ψ will not be removable. However, it may be verified that *all the invariants and covariants of the surface referred to its asymptotic curves are expressible explicitly in terms of the coefficients and variables which appear in equations* (16). The quantities ϕ

* Darboux, "Théorie Générale des Surfaces," t. I, p. 145. The above transformation is essentially that made by Wilczynski in his first memoir on curved surfaces, *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 243.

and ψ will be found to occur only as first derivatives ϕ_v and ψ_v, but always as extraneous factors.*

The quantities γ and β' are invariants. If we discard the factors outside the brackets, and write $\bar{\gamma}$, $\bar{\beta}'$ for the brackets themselves, we find that

$$\bar{\gamma} = \mathfrak{C}' + \sqrt{-a}\,\mathfrak{B}', \quad \bar{\beta}' = \mathfrak{C}' - \sqrt{-a}\,\mathfrak{B}', \tag{63}$$

where $\mathfrak{B}'$ and $\mathfrak{C}'$ are two of the fundamental invariants (38).

We are now able to give the geometric significance of these invariants. If, in the equations

$$\left.\begin{aligned} y_{uu} &= \beta\, y_u + \gamma\, y_v + \delta\, y, \\ y_{vv} &= \beta'\, y_u + \gamma'\, y_v + \delta'\, y, \end{aligned}\right\} \tag{64}$$

the coefficient γ is zero, then the curves v=const. are straight lines, and S_0 is a ruled surface. Similarly, if $\beta'=0$, the curves u=const. are straight lines. If both γ and β' are zero, the surface is a quadric, since quadrics are the only ruled surfaces having two families of straight-line generators. We may say, then, that *the surface S_0 is ruled if and only if the invariant*

$$\bar{\beta}'\,\bar{\gamma} = a\,\mathfrak{B}'^2 + \mathfrak{C}'^2$$

is zero. If $\bar{\beta}'$ and $\bar{\gamma}$ are both zero, that is, if $\mathfrak{B}'$ and $\mathfrak{C}'$ are both zero, the surface is a quadric.

We shall not consider in this paper the many problems which here suggest themselves. In particular, it would be important to put into relation with the various congruences and ruled surfaces associated with the asymptotic net certain similar configurations connected with the conjugate net which we have been considering. The nature of the surface in the neighborhood of a point has been studied in detail by Wilczynski in the five memoirs on curved surfaces already quoted, and the bearing of these results on the conjugate net of curves has still to be investigated. We leave these and similar questions open for the present.

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* These remarks, and the similar ones at the end of § 5, are important, though they seem to have escaped notice until now. In many of the problems in projective differential geometry already studied by Wilczynski and others, it has been found convenient to reduce the differential equations to certain simplified forms, or, we should say, to suppose them to be so reduced. This reduction generally requires the integration of differential equations, which of course is generally impossible. Nevertheless, it will be found, as we have already seen, that *if the reduced form bears an intrinsic geometric relation to the original form,* then the invariants and covariants of the reduced form are expressible in terms of the coefficients of the original form, without the integration of any differential equations. This remark has in fact enabled the present writer to set up a complete system of invariants for the general theory of curved surfaces. Cf. G. M. Green, "On the Theory of Curved Surfaces, and Canonical Systems in Projective Differential Geometry," *Transactions of the American Mathematical Society,* Vol. XVI (1915), pp. 1–12.

§ 8. *The Differential Equations for the General Theory of Congruences of Straight Lines.*

In his prize memoir* of 1909, Wilczynski developed a projective theory of congruences of straight lines, basing it upon the consideration of a completely integrable system of partial differential equations. Evidently, the theory of a one-parameter family of curves is geometrically equivalent to the theory of the congruence of tangents to these curves, so that it will be of interest to connect the two theories analytically by setting up the completely integrable system for the theory of congruences, and relating it to the system of equations (16). Wilczynski, however, assumes the focal surface of the congruence to be known. Since in a later development we shall need the equations for the more general case, we shall set up the completely integrable system for a congruence neither sheet of whose focal surface is known.

Let the congruence consist of the ∞^2 lines joining corresponding points† on the surfaces S_y and S_z given by the equations

$$y^{(k)} = f^{(k)}(u, v), \quad z^{(k)} = g^{(k)}(u, v), \qquad (k = 1, 2, 3, 4). \tag{65}$$

These surfaces may evidently always be chosen so that the determinant

$$\Delta = \begin{vmatrix} y^{(1)} & z^{(1)} & y_u^{(1)} & z_v^{(1)} \\ y^{(2)} & z^{(2)} & y_u^{(2)} & z_v^{(2)} \\ y^{(3)} & z^{(3)} & y_u^{(3)} & z_v^{(3)} \\ y^{(4)} & z^{(4)} & y_u^{(4)} & z_v^{(4)} \end{vmatrix} \tag{66}$$

is not identically zero. Then we may always determine the coefficients in the partial differential equations

$$\begin{aligned} y_v &= a^{(1)} y + b^{(1)} z + c^{(1)} y_u + d^{(1)} z_v, \\ z_u &= a^{(2)} y + b^{(2)} z + c^{(2)} y_u + d^{(2)} z_v, \\ y_{uu} &= \alpha^{(1)} y + \beta^{(1)} z + \gamma^{(1)} y_u + \delta^{(1)} z_v, \\ z_{vv} &= \alpha^{(2)} y + \beta^{(2)} z + \gamma^{(2)} y_u + \delta^{(2)} z_v, \end{aligned} \tag{67}$$

so that the quantities (65) may be solutions of (67), just as the coefficients of system (4) were found in § 1. System (67) is the completely integrable system we seek; conversely, *any* system of form (67), whose coefficients satisfy suitable integrability conditions, will be completely integrable and may be taken to define two surfaces (65) and the associated congruence. The most general system of solutions of form (65) will be a projective transformation of any particular fundamental system of solutions, so that we have to study the invariants and covariants of (67) under the most general transformations of the variables which leave fixed the congruence of lines. The transformation

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* "Sur la Théorie Générale des Congruences," *Académie Royale de Belgique, Classe des Sciences,* Mémoires in 4°, 2. série, t. III (1910–1912).

† *I. e.,* points corresponding to the same values of the parameters u, v.

$$\begin{aligned} y &= \alpha \bar{y} + \beta \bar{z}, \\ z &= \gamma \bar{y} + \delta \bar{z}, \end{aligned} \qquad (\alpha\delta - \beta\gamma \neq 0), \tag{68}$$

where α, β, γ, δ are functions of u and v, is the most general transformation which leaves fixed each line of the congruence, but changes the surfaces S_y, S_z in an arbitrary way. Similarly, the transformation

$$\bar{u} = U(u, v), \quad \bar{v} = V(u, v) \tag{69}$$

interchanges among themselves the lines of the congruence. Transformations (68) and (69) together form the most general group of point transformations which leave fixed the congruence of lines.

We now seek a transformation of form (68) which changes the surfaces S_y and S_z into the two sheets of the focal surface of the congruence. Substituting (68) in the first two of equations (67), we obtain two equations in the new dependent variables $\bar{y}$, $\bar{z}$. Solving these for $\bar{y}_v$, $\bar{z}_u$, we obtain two equations of the same form as the first two of (67), and for which the new coefficients $\bar{d}^{(1)}$ and $\bar{c}^{(2)}$ have the values

$$\left.\begin{aligned} \bar{d}^{(1)} &= \frac{c^{(2)}\beta^2 + (c^{(1)}d^{(2)} - d^{(1)}c^{(2)} - 1)\beta\delta + d^{(1)}\delta^2}{(\alpha - d^{(1)}\gamma)(\delta - c^{(2)}\beta) - c^{(1)}d^{(2)}\beta\gamma}, \\ \bar{c}^{(2)} &= \frac{c^{(2)}\alpha^2 + (c^{(1)}d^{(2)} - d^{(1)}c^{(2)} - 1)\alpha\gamma + d^{(1)}\gamma^2}{(\alpha - d^{(1)}\gamma)(\delta - c^{(2)}\beta) - c^{(1)}d^{(2)}\beta\gamma}. \end{aligned}\right\} \tag{70}$$

We can therefore make $\bar{d}^{(1)}$ and $\bar{c}^{(2)}$ both vanish by taking for β/δ and α/γ the two roots of the quadratic

$$c^{(2)}t^2 + (c^{(1)}d^{(2)} - d^{(1)}c^{(2)} - 1)t + d^{(1)} = 0, \tag{71}$$

provided the roots be distinct. The differential equations then take the forms

$$\begin{aligned} \bar{y}_v &= \bar{a}^{(1)}\bar{y} + \bar{b}^{(1)}\bar{z} + \bar{c}^{(1)}\bar{y}_u, \\ \bar{z}_u &= \bar{a}^{(2)}\bar{y} + \bar{b}^{(2)}\bar{z} + \bar{d}^{(2)}\bar{z}_v, \end{aligned}$$

which show that the line $\bar{y}\bar{z}$ is tangent to the surfaces $S_{\bar{y}}$ and $S_{\bar{z}}$. From (68) we have

$$\bar{y} = \frac{\delta y - \beta z}{\alpha\delta - \beta\gamma}, \quad \bar{z} = \frac{-\gamma y + \alpha z}{\alpha\delta - \beta\gamma}. \tag{72}$$

Therefore, *the two sheets of the focal surface of the congruence are given by* (72), *in which β/δ and α/γ are the roots* (*supposed distinct*) *of the quadratic* (71).

If the discriminant of the quadratic (71) vanishes, it is easily seen that the two sheets of the focal surface coincide. This being impossible for the congruence of tangents to our one-parameter family of curves, we shall discard the case of coincident roots.

We turn now to a certain congruence associated with our family of curves C_u. If we join the points ρ and σ corresponding to a point y, and do the same for every point y of the surface S_0, we obtain a congruence of lines, whose focal surface we proceed to determine. With the aid of (52) and (53) it is easily found that

$$\left.\begin{aligned} \rho_v &= a^{(1)}\rho + b^{(1)}\sigma + c^{(1)}\rho_u + d^{(1)}\sigma_v, \\ \sigma_u &= a^{(2)}\rho + b^{(2)}\sigma + c^{(2)}\rho_u + d^{(2)}\sigma_v, \end{aligned}\right\} \tag{73}$$

where

$$\left.\begin{aligned} &a^{(1)}=b'-\frac{K}{\mathfrak{D}}(b-c'), \quad b^{(1)}=-\frac{K}{\mathfrak{D}}(ab'+c), \quad c^{(1)}=\frac{K}{\mathfrak{D}}, \quad d^{(1)}=-\frac{K}{\mathfrak{D}}a, \\ &a^{(2)}=-\frac{H}{\mathfrak{D}}(b-c'), \quad b^{(2)}=c'-\frac{H}{\mathfrak{D}}(ab'+c), \quad c^{(2)}=\frac{H}{\mathfrak{D}}, \quad d^{(2)}=-\frac{H}{\mathfrak{D}}a. \end{aligned}\right\} \tag{74}$$

The quadratic (71) is for this case

$$H t^2 - \mathfrak{D} t - K A = 0,$$

whose roots are

$$t_1, t_2 = \frac{\mathfrak{D} \pm \sqrt{\mathfrak{D}^2 + 4AHK}}{2H}.$$

Substituting these roots for β/δ, α/γ in (72), and discarding certain factors, *we find the covariants*

$$\left.\begin{aligned} R &= 2H\rho - (\mathfrak{D} + \sqrt{\mathfrak{D}^2+4AHK})\,\sigma, \\ S &= 2H\rho - (\mathfrak{D} - \sqrt{\mathfrak{D}^2+4AHK})\,\sigma, \end{aligned}\right\} \tag{75}$$

which give the two sheets of the focal surface of the congruence of lines joining corresponding points ρ and σ.

The four points ρ, σ, R, S lie on a line; their anharmonic ratio must be an absolute invariant. It is, in fact,

$$\frac{\left(\dfrac{\mathfrak{D}+\sqrt{\mathfrak{D}^2+4AHK}}{2H}\right)(\infty)}{\left(\dfrac{\mathfrak{D}-\sqrt{\mathfrak{D}^2+4AHK}}{2H}\right)(\infty)} = -1 - \frac{\mathfrak{D}}{2AHK}(\mathfrak{D} + \sqrt{\mathfrak{D}^2+4AHK}).$$

In the above it has been supposed that $\mathfrak{D} \neq 0$. But taking the quantities (75) as they stand, regardless of the method of their derivation, we easily find that they actually satisfy two equations of the form

$$\begin{aligned} R_v &= a^{(1)}R + b^{(1)}S + c^{(1)}R_u, \\ S_u &= a^{(2)}R + b^{(2)}S + d^{(2)}S_v, \end{aligned}$$

and hence give the two sheets of the focal surface, provided only that $\mathfrak{D}$ and H do not both vanish. We may then state the following simple theorems in regard to the congruence of lines joining corresponding points ρ, σ:

The sheets of the focal surface coincide if $\mathfrak{D}^2 + 4\,AHK = 0$.

On every line $\rho\sigma$ *of the congruence, the pair of points* ρ, σ *is separated harmonically by the pair of focal points* R, S *if and only if* $\mathfrak{D} = 0$.

This last gives a geometric interpretation for the vanishing of the fundamental invariant $\mathfrak{D}$.

We return now to the general theory of congruences. The surfaces S_y, S_z have been transformed into the two sheets of the focal surface of the congruence, but the differential equations are not yet in the form used by Wilczynski. By a suitable transformation of the form

$$\bar{u} = U\,(u, v), \quad \bar{v} = V\,(u, v), \tag{69}$$

the differential equations of the first order may be transformed into

$$y_v + \alpha\, y = \omega\, z, \quad z_u + \beta\, z = \omega'\, y. \tag{76}$$

We shall not carry out this transformation, which is equivalent to the determination of the two systems of developables of the congruence, and requires the integration of a partial differential equation of the first order. These developables are known for the congruences to which we intend to apply Wilczynski's results.

We consider now the congruence of tangents to the one-parameter family of curves C_u. The two sheets of the focal surface are given by the covariants ρ and y, which we know satisfy the differential equations

$$\rho_v - b'\rho = K\,y, \quad y_u - c'\,y = \rho, \tag{77}$$

which are of the form (76). Instead of ρ and y we shall take as dependent variables the quantities

$$\eta = \lambda\,\rho, \quad \zeta = \mu\,y, \tag{78}$$

where λ and μ satisfy the relations

$$\lambda_v + b'\,\lambda = 0, \quad \mu_u + c'\,\mu = 0. \tag{79}$$

The new variables are easily seen to satisfy the differential equations

$$\eta_v = m\,\zeta, \quad \zeta_u = n\,\eta, \tag{80}$$

where

$$m = \frac{\lambda}{\mu} K, \quad n = \frac{\mu}{\lambda}. \tag{80a}$$

But the completely integrable system of differential equations contains besides (80) two equations of the second order satisfied by η and ζ. These equations are found most easily as follows. We have from (52)

$$a\,y_{vv} = \rho_u - (b - c')\,\rho - c\,y_v - [\,d - c'_u + c'\,(b - c')\,]\,y,$$

and find also that

$$\begin{aligned}\rho_{uu} = {} & (\alpha^{(1)} - a\,c')\,y_{vv} + (\beta^{(1)} - 2\,c'_u - b\,c')\,y_u \\ & + (\gamma^{(1)} - c\,c')\,y_v + (\delta^{(1)} - c'_{uu} - d\,c')\,y,\end{aligned}$$

$\alpha^{(1)}$, $\beta^{(1)}$, $\gamma^{(1)}$, $\delta^{(1)}$ being given by (21). The first of these may be transformed immediately by means of (78) into the last equation of system (81) below. The expression for ρ_{uu} may be transformed likewise into the third equation of system (81), after y_{vv} and y_u have been replaced by equivalent expressions in ρ_u, ρ, y_v, y. We thus obtain, after some calculation, the last two of the following completely integrable system of differential equations:

$$\left.\begin{aligned} \eta_v &= m\zeta, \quad \zeta_u = n\eta, \\ \eta_{uu} &= \alpha\eta + \beta\zeta + \gamma\eta_u + \delta\zeta_v, \\ \zeta_{vv} &= \alpha'\eta + \beta'\zeta + \gamma'\eta_u + \delta'\zeta_v, \end{aligned}\right\} \tag{81}$$

where

$$\left.\begin{aligned} m &= \frac{\lambda}{\mu}K, & n &= \frac{\mu}{\lambda}, \\ \gamma &= b + \frac{a_u}{a} + 2\frac{\lambda_u}{\lambda}, & \delta &= \frac{\lambda}{\mu}\left[\gamma^{(1)} - cc' - c\left(b + \frac{a_u}{a}\right)\right], \\ \gamma' &= \frac{\mu}{\lambda}\frac{1}{a}, & \delta' &= -\frac{c}{a} + 2\frac{\mu_v}{\mu}, \end{aligned}\right\} \tag{81a}$$

and the values of α, β, α', β' do not concern us.

System (81) is the completely integrable system (D) which Wilczynski takes as the basis for his theory of congruences. To avoid confusion in notation, we have changed all of Wilczynski's letters, except u, v, m and n, into the corresponding Greek letters.

We find very easily, by using the expression for $\gamma^{(1)}$ from (21), that

$$\left.\begin{aligned} \delta &= \frac{\lambda}{\mu}\left[\gamma^{(1)} - cc' - c\left(b + \frac{a_u}{a}\right)\right] \\ &= \frac{\lambda}{\mu}\left[c_u + ac'_v + ad' + ab'c' - \frac{ca_u}{a}\right] \\ &= A\frac{\lambda}{\mu}[H + 3\mathfrak{B}'_u + \mathfrak{C}'_v] \\ &= A\frac{\lambda}{\mu}\left[K + 2(B'_u + C'_v) - \frac{\partial^2}{\partial u\,\partial v}\log A\right]. \end{aligned}\right\} \tag{81b}$$

Except for the factor $\frac{\lambda}{\mu}$, this is an invariant of the one-parameter family of curves.

It is readily verified that

$$\gamma_v = \delta'_u. \tag{82}$$

In fact, on reduction this is found to be equivalent to the relation

$$\frac{\partial}{\partial v}(b + 2c') = \frac{\partial}{\partial u}\left(\frac{2ab' - c - a_v}{a}\right).$$

We may therefore put, with Wilczynski,

$$f_u = \gamma, \quad f_v = \delta'. \tag{83}$$

Then the four quantities $\overset{(\eta)}{\mathfrak{B}}, \overset{(\eta)}{\mathfrak{C}}'', \overset{(\zeta)}{\mathfrak{B}}, \overset{(\zeta)}{\mathfrak{C}}''$ given by

$$\left.\begin{aligned} 4\overset{(\eta)}{\mathfrak{B}} &= f_u - \frac{1}{2}\frac{\delta_u}{\delta} - \frac{3}{2}\frac{m_u}{m}, & 4\overset{(\eta)}{\mathfrak{C}}'' &= f_v + \frac{3}{2}\frac{\delta_v}{\delta} + \frac{1}{2}\frac{m_v}{m}, \\ 4\overset{(\zeta)}{\mathfrak{B}} &= f_u + \frac{3}{2}\frac{\gamma'_u}{\gamma'} + \frac{1}{2}\frac{n_u}{n}, & 4\overset{(\zeta)}{\mathfrak{C}}'' &= f_v - \frac{1}{2}\frac{\gamma'_v}{\gamma'} - \frac{3}{2}\frac{n_v}{n} \end{aligned}\right\} \tag{84}$$

are invariants of the congruence, and these, with the invariants m, n, γ', δ, form a system of eight invariants such that any six may be taken as a fundamental system in the sense that any invariant is a function of these six and of their derivatives.*

The quantities (84) are invariants of the one-parameter family of curves, and are expressible in terms of the coefficients of (16). Upon calculation we find in fact that

$$\left.\begin{aligned} 4\overset{(\eta)}{\mathfrak{B}} &= -4\mathfrak{C}' + \frac{A_u}{A} - \frac{3}{2}\frac{K_u}{K} - \frac{1}{2}\frac{H_u + 3\mathfrak{B}'_{uu} + \mathfrak{C}'_{uv}}{H + 3\mathfrak{B}'_u + \mathfrak{C}'_v}, \\ 4\overset{(\eta)}{\mathfrak{C}}'' &= -4\mathfrak{B}' + \frac{A_v}{A} + \frac{1}{2}\frac{K_v}{K} + \frac{3}{2}\frac{H_v + 3\mathfrak{B}'_{uv} + \mathfrak{C}'_{vv}}{H + 3\mathfrak{B}'_u + \mathfrak{C}'_v}, \\ 4\overset{(\zeta)}{\mathfrak{B}} &= -4\mathfrak{C}', \quad 4\overset{(\zeta)}{\mathfrak{C}}'' = -4\mathfrak{B}'. \end{aligned}\right\} \tag{85}$$

We may choose as the six fundamental invariants these four and

$$m n = K, \qquad \gamma'\delta = H + 3\mathfrak{B}'_u + \mathfrak{C}'_v.$$

Then it is immediately evident that all invariants of the congruence, being functions of these six and of their derivatives, are expressible in terms of the coefficients of (16) and their derivatives, the knowledge of integrals of equations (79) being unnecessary. Any invariant equation of the theory of congruences will therefore express a property of a one-parameter family of curves. Thus, *the vanishing of the invariant*

$$\left.\begin{aligned} W &= m n - \gamma'\delta \\ &= K - (H + 3\mathfrak{B}'_u + \mathfrak{C}'_v) \\ &= -2(B'_u + C'_v) + \frac{\partial^2}{\partial u\,\partial v}\log A \end{aligned}\right\} \tag{86}$$

is the condition that the congruence of tangents to our one-parameter family of curves be a W-congruence, i. e., a congruence in which asymptotic lines on the two sheets of the focal surface correspond.†

Again, we may express the conditions that the congruence of tangents to the one-parameter family of curves belong to a linear complex, or, as we may say, the conditions that the one-parameter family of curves belong to a linear

* E. J. Wilczynski, *loc. cit.*, pp. 20, 23.

† E. J. Wilczynski, *loc. cit.*, p. 46.

complex. The conditions, as given by equations (73a) of Wilczynski's memoir, are

$$\left.\begin{array}{l} W=0, \quad \gamma'\beta+mn_v=0, \quad \alpha' m+\gamma' m_u=0, \\ \begin{vmatrix} m_u-m\gamma & \gamma'_u \\ m & \gamma' \end{vmatrix}=0, \quad \begin{vmatrix} m_v & \gamma'_v-\gamma'\delta' \\ m & \gamma' \end{vmatrix}=0. \end{array}\right\} \tag{87}$$

But by the integrability conditions (12) of that memoir we have

$$\beta=-\delta_v-\delta f_v, \quad \alpha'=-\gamma'_u-\gamma' f_u,$$

so that the second and third of equations (87) become

$$m n_v-\gamma'\delta_v-\gamma'\delta\delta'=0, \quad \gamma' m_u-m\gamma'_u-m\gamma\gamma'=0,$$

or, making use of the relation

$$W\equiv mn-\gamma'\delta=0,$$

$$m\gamma'_v-m_v\gamma'-m\gamma'\delta'=0, \quad \gamma' m_u-m\gamma'_u-m\gamma\gamma'=0. \tag{88}$$

These are exactly equivalent to the fifth and fourth, respectively, of equations (87). Equations (88) are invariant equations, and when expressed in terms of the coefficients of (16) they become, after a somewhat lengthy reduction,

$$\left.\begin{array}{l} 4\mathfrak{B}'-\dfrac{K_v}{K}-\dfrac{A_v}{2A}=0, \\ 4\mathfrak{C}'+\dfrac{K_u}{K}-\dfrac{A_u}{2A}=0. \end{array}\right\} \tag{89}$$

If the first of these be differentiated with respect to u, the second with respect to v, and the results added, we obtain the equation

$$4(\mathfrak{B}'_u+\mathfrak{C}'_v)-\frac{\partial^2}{\partial u\,\partial v}\log A=0,$$

which is the same as $2W=0$. We have then the theorem: *The one-parameter family of curves belongs to a linear complex if and only if*

$$4\mathfrak{B}'-\frac{K_v}{K}-\frac{A_v}{2A}=0, \quad 4\mathfrak{C}'+\frac{K_u}{K}-\frac{A_u}{2A}=0.$$

Enough has been indicated to show how the results of Wilczynski's theory of congruences may be applied directly to the study of a one-parameter family of curves. In fact, the two theories are practically identical geometrically, so that either may be used to approach the theory of congruences. Which is to be employed in any particular case must be decided by considerations of economy in calculation.

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